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Key Aspects of Eigenvectors in Assorted Domains

Parmila Kumari

Research Scholar

Shri Venkateshwara University

Gajraula, U.P., India

Dr. Amardeep Singh

Assistant Professor

Shri Venkateshwara University

Gajraula, U.P., India

Abstract

Eigenvectors and eigenvalues have many important applications in computer vision and machine learning in general. Well known examples are PCA (Principal Component Analysis) for dimensionality reduction or EigenFaces for face recognition. Eigenvalues and eigenvectors feature prominently in the analysis of linear transformations. The prefix *eigen-* is adopted from the German word *eigen* for "proper", "characteristic". Originally utilized to study principal axes of the rotational motion of rigid bodies, eigenvalues and eigenvectors have a wide range of applications, for example in stability analysis, vibration analysis, atomic orbitals, facial recognition, and matrix diagonalization. In essence, an eigenvector **v** of a linear transformation T is a non-zero vector that, when T is applied to it, does not change direction. Applying T to the eigenvector only scales the eigenvector by the scalar value λ, called an eigenvalue.

Keywords: Eigenvector, Eigenvalues, Features of Eigenvectors

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Introduction

If the vector space V is finite-dimensional, then the linear transformation T can be represented as a square matrix A, and the vector **v** by a column vector, rendering the above mapping as a matrix multiplication on the left-hand side and a scaling of the column vector on the right-hand side in the equation

There is a direct correspondence between n-by-n square matrices and linear transformations from an n-dimensional vector space to itself, given any basis of the vector space. For this reason, it is equivalent to define eigenvalues and eigenvectors using either the language of matrices or the language of linear transformations.

Geometrically, an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction that is stretched by the transformation and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Find the eigenvalues of A and their associated eigenvectors.

[solution:]

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigenvector associated with the eigenvalue λ. Then,

$$Ax = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda x = (\lambda I)x \quad \Leftrightarrow \quad (\lambda I)x - Ax = (\lambda I - A)x = 0.$$

Thus,

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the nonzero (nontrivial) solution of the homogeneous linear system

$(\lambda I - A)x = 0$. $\Leftrightarrow$ $\lambda I - A$ is singular $\Leftrightarrow$ $\det(\lambda I - A) = 0$.

Therefore,

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$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = (\lambda - 3)(\lambda - 2) = 0$$

$$\Leftrightarrow \quad \lambda = 2 \ \text{ or } 3.$$

1. As $\lambda = 2$,

$$Ax = 2x = 2Ix \Leftrightarrow 2Ix - Ax = (2I - A)x = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 .$$

$$\Leftrightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t, \quad t \in R.$$

$$\Leftrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} t, \quad t \neq 0, \ t \in R, \text{ are the eigenvectors associated with } \lambda = 2$$

2. As $\lambda = 3$,

$$Ax = 3x = 3Ix \Leftrightarrow 3Ix - Ax = (3I - A)x = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 .$$

$$\Leftrightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} r, \quad r \in R.$$

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$$\Leftrightarrow \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} r, \quad r \neq 0, \; r \in R, \text{ are the eigenvectors associated with } \lambda = 3$$

Note:

In the above example, the eigenvalues of *A* satisfy the following equation

$$\det(\lambda I - A) = 0 .$$

After finding the eigenvalues, we can further solve the associated homogeneous system to find the eigenvectors.

Definition of the characteristic polynomial:

Let $A_{n \times n} = \lfloor a_{ij} \rfloor$**. The determinant**

$$f(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix},$$

is called the characteristic polynomial of *A*.

$$f(\lambda) = \det(\lambda I - A) = 0,$$

is called the characteristic equation of *A*.

A is singular if and only if 0 is an eigenvalue of A.

[proof:]

$\Rightarrow$:.

A is singular $\Rightarrow$ $Ax = 0$ has non-trivial solution $\Rightarrow$ There exists a nonzero vector ***x*** such that

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$$Ax = 0 = 0x.$$

$\Rightarrow$ ***x*** is the eigenvector of ***A*** associated with eigenvalue 0.

$\Leftarrow$:

0 is an eigenvalue of ***A*** $\Rightarrow$ There exists a nonzero vector ***x*** such that

$$Ax = 0 = 0x.$$

$\Rightarrow$ The homogeneous system $Ax = 0$ has nontrivial (nonzero) solution.

$\Rightarrow$ ***A*** is singular.

The eigenvalues of A are the real roots of the characteristic polynomial of A.

$\Rightarrow$:

Let λ^* be an eigenvalue of ***A*** associated with eigenvector ***u.*** Also, let $f(\lambda)$ be the characteristic polynomial of ***A.*** Then,

$Au = \lambda^* u \quad \Rightarrow \quad \lambda^* u - Au = \lambda^* Iu - Au = (\lambda^* I - A)u = 0 \quad \Rightarrow$ The homogeneous system has nontrivial (nonzero) solution x $\Rightarrow$ $\lambda^* I - A$ is singular $\Rightarrow$

$$\det(\lambda^* I - A) = f(\lambda^*) = 0.$$

$\Rightarrow$ λ^* is a real root of $f(\lambda) = 0$.

$\Leftarrow$:

Let λ_r be a real root of $f(\lambda) = 0 \Rightarrow f(\lambda_r) = \det(\lambda_r I - A) = 0 \Rightarrow \lambda_r I - A$ is a singular matrix $\Rightarrow$ There exists a nonzero vector (nontrivial solution) v such that

$$(\lambda_r I - A)v = 0 \Rightarrow Av = \lambda_r v.$$

$\Rightarrow$ v is the eigenvector of A associated with the eigenvalue λ_r.

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Procedure of finding the eigenvalues and eigenvectors of A:

1. **Solve for the real roots of the characteristic equation** $f(\lambda)=0$. **These real roots** $\lambda_1, \lambda_2, \ldots$ **are the eigenvalues of** A.

2. **Solve for the homogeneous system** $(A-\lambda_i I)x=0$ **or** $(\lambda_i I - A)x = 0$, $i=1,2,\ldots$. **The nontrivial (nonzero) solutions are the eigenvectors associated with the eigenvalues** λ_i.

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

$$f(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-5 & -4 & -2 \\ -4 & \lambda-5 & -2 \\ -2 & -2 & \lambda-2 \end{vmatrix} = (\lambda-1)^2(\lambda-10) = 0$$

$\Rightarrow \lambda = 1, 1,$ and 10.

1. As $\lambda = 1$,

$$(1 \cdot I - A)x = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\Leftrightarrow x_1 = -s-t,\ x_2 = s,\ x_3 = 2t \ \Leftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ 2t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\ s,t \in R.$$

Thus,

$$s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\2\end{bmatrix},\ s,t\in R,\ s\neq 0 \text{ or } t\neq 0,$$

are the eigenvectors associated with eigenvalue $\lambda = 1$.

2. As $\lambda = 10$,

$$(10\cdot I - A)x = \begin{bmatrix}5 & -4 & -2\\-4 & 5 & -2\\-2 & -2 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0.$$

$$\Leftrightarrow x_1 = 2r,\ x_2 = 2r,\ x_3 = r \ \Leftrightarrow x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2r\\2r\\r\end{bmatrix} = r\begin{bmatrix}2\\2\\1\end{bmatrix},\ r\in R.$$

Thus,

$$r\begin{bmatrix}2\\2\\1\end{bmatrix},\ r\in R,\ r\neq 0,$$

are the eigenvectors associated with eigenvalue $\lambda = 10$.

$$A = \begin{bmatrix}0 & 1 & 2\\2 & 3 & 0\\0 & 4 & 5\end{bmatrix}.$$

Find the eigenvalues and the eigenvectors of ***A*.**

$$f(\lambda) = \det(\lambda I - A) = \begin{vmatrix}\lambda & -1 & -2\\-2 & \lambda-3 & 0\\0 & -4 & \lambda-5\end{vmatrix} = (\lambda-1)^2(\lambda-6) = 0$$

$\Rightarrow \lambda = 1, 1,$ and 6.

3. As $\lambda = 1$,

$$\left(A - 1 \cdot I\right)x = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 2 & 0 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 .$$

Conclusion

The popular periodic table layout, also known as the common or standard form (as shown at various other points in this article), is attributable to Horace Groves Deming. In 1923, Deming, an American chemist, published short (Mendeleev style) and medium (18-column) form periodic tables. Merck and Company prepared a handout form of Deming's 18-column medium table, in 1928, which was widely circulated in American schools. By the 1930s Deming's table was appearing in handbooks and encyclopedias of chemistry. It was also distributed for many years by the Sargent-Welch Scientific Company. With the development of modern quantum mechanical theories of electron configurations within atoms, it became apparent that each period (row) in the table corresponded to the filling of a quantum shell of electrons. Larger atoms have more electron sub-shells, so later tables have required progressively longer periods.

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